

Analytic Loss Minimization: A Proof

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Abstract—Loss minimizing generator dispatch profiles for power systems are usually derived using optimization techniques. However, some authors have noted that a system's K_{GL} matrix can be used to analytically determine a loss minimizing dispatch. This letter draws on recent research on the characterization of transmission system losses to demonstrate how the K_{GL} matrix achieves this. A new proof of the observed zero row summation property of the Y_{GGM} matrix is provided to this end.

I. INTRODUCTION

Various works [1], [2] have noted that the K_{GL} matrix allows the direct calculation of a generator dispatch profile that appears to minimize active power losses. The literature offers scant explanation for how this result is achieved. The recent work of Abdelkader et al. [3], [4] offers useful insight here. Crucially, [4] shows how network losses can be separated into three distinct components. One of these loss components arises solely because of mismatched generator voltages, which cause circulating currents to flow through the system [3]. This letter demonstrates how the K_{GL} matrix can give an ideal generator dispatch profile which equalizes generator complex voltages, so no circulating currents will flow and this loss component is nullified.

II. SEPARATION OF SYSTEM LOSSES

A. Partitioning

The Y_{bus} is reordered, per [5], such that the m generator buses (G subscript) and n load buses (L) are grouped together:

$$\begin{bmatrix} I_G \\ I_L \end{bmatrix} = \begin{bmatrix} Y_{GG} & Y_{GL} \\ Y_{LG} & Y_{LL} \end{bmatrix} \begin{bmatrix} V_G \\ V_L \end{bmatrix} \quad (1)$$

I_G and I_L are complex-valued vectors representing the nodal currents at generator and load buses, respectively, while V_G and V_L are corresponding complex nodal voltages. Manipulation of (1) gives:

$$\begin{bmatrix} V_L \\ I_G \end{bmatrix} = \begin{bmatrix} Z_{LL} & F_{LG} \\ K_{GL} & Y_{GGM} \end{bmatrix} \begin{bmatrix} I_L \\ V_G \end{bmatrix} \quad (2)$$

Where:

$$Y_{GGM} = Y_{GG} - Y_{GL}Z_{LL}Y_{LG} \quad (3)$$

$$F_{LG} = -Z_{LL}Y_{LG} = -K_{GL} \quad (4)$$

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B. Loss Characterization

Per [4], the network loss is calculated as the difference between aggregate powers on the generation and load sides of the network. To find the total power generated, use the I_G expression from (2), left-multiplied by V_G to give a single complex value:

$$S_G^{Tot} = V_G^T I_G^* = V_G^T Y_{GGM}^* V_G^* + V_G^T Y_{GL}^* Z_{LL}^* I_L^* \quad (5)$$

Equivalently, multiplying the V_L expression from (2) by I_L expresses the total load power consumed as a complex number:

$$S_L^{Tot} = V_L^T I_L^* = I_L^T Z_{LL}^T I_L^* - V_G^T Y_{LG}^T Z_{LL}^T I_L^* \quad (6)$$

Given their opposite signs, summing these expressions gives the total system loss:

$$S_{Loss}^{Tot} = S_G^{Tot} + S_L^{Tot} \quad (7)$$

Which equals:

$$V_G^T Y_{GGM}^* V_G^* + V_G^T (Y_{GL}^* Z_{LL}^* - Y_{LG}^T Z_{LL}^T) I_L^* + I_L^T Z_{LL}^T I_L^* \quad (8)$$

The system losses naturally separate themselves into three distinct components. The first term is the circulating current loss, and it depends solely on generator voltages. The middle component, the mismatch loss, will only arise when branch X/R ratios are heterogeneous, and will in any case generally be small and imaginary [4]. The last component is the load current loss, which cannot be affected by the system operator, being a pure consequence of system topology and load currents. As will be shown, the Y_{GGM} matrix has the property that its rows sum to zero, and so its product can be brought to zero by right-multiplying it by a vector whose elements are homogeneous. Thus, if generator voltages are uniform, there will be no excess circulating current loss, and the active power losses will reduce to the irreducible component which arises from serving load currents.

III. ZERO ROW SUMMATION OF Y_{GGM}

Theorem If every row of the Y_{bus} matrix sums to zero and $\det Y_{LL} \neq 0$, then every row of Y_{GGM} sums to zero. If every row of the Y_{bus} matrix sums approximately to zero, or if it sums to zero but $\det Y_{LL} = 0$, then every row of the matrix Y_{GGM} sums approximately to zero.

Proof The matrix Y_{GGM} can be written as

$$Y_{GGM} = Y_{GG} + Y_{GL}F_{LG}, \quad (9)$$

since $F_{LG} = -Z_{LL}Y_{LG}$. Where

$$Z_{LL} = \begin{cases} Y_{LL}^{-1}, & \det(Y_{LL}) \neq 0 \\ Y_{LL}^\dagger, & \det(Y_{LL}) = 0 \end{cases} \quad (10)$$

For ease of notation set $Y_{GG} = [a_{ij}]_{i=1,2,\dots,m}^{j=1,2,\dots,m}$, and similarly:

$$\begin{aligned} Y_{GL} &= [b_{ij}]_{m \times n} & F_{LG} &= [c_{ij}]_{n \times m} \\ Y_{GL}F_{LG} &= [d_{ij}]_{m \times m} & Y_{GGM} &= [g_{ij}]_{m \times m} \end{aligned} \quad (11)$$

By substituting the previous expressions into (9), for every row $i = 1, 2, \dots, m$ we have:

$$\sum_{j=1}^m g_{ij} = \sum_{j=1}^m a_{ij} + \sum_{j=1}^m d_{ij}. \quad (12)$$

In addition:

$$\begin{aligned} d_{i1} &= \sum_{k=1}^n b_{ik}c_{k1} \\ &\vdots \\ d_{im} &= \sum_{k=1}^n b_{ik}c_{km}. \end{aligned}$$

By taking the sum of the above equalities, $\forall i = 1, 2, \dots, n$ we arrive at:

$$\sum_{j=1}^m d_{ij} = \sum_{j=1}^m \sum_{k=1}^n b_{ik}c_{kj}. \quad (13)$$

Disregarding shunt elements, the Y_{bus} is a weighted Laplacian matrix [6], and so each of its rows will sum to zero. Accordingly:

$$\sum_{j=1}^m a_{ij} + \sum_{j=1}^n b_{ij} = 0, \quad \forall i = 1, 2, \dots, m. \quad (14)$$

By replacing (13) into (12) we get:

$$\sum_{j=1}^m g_{ij} = \sum_{j=1}^m a_{ij} + \sum_{j=1}^m \sum_{k=1}^n b_{ik}c_{kj} \quad (15)$$

and using (14):

$$\sum_{j=1}^m g_{ij} = -\sum_{k=1}^n b_{ik} + \sum_{j=1}^m \sum_{k=1}^n b_{ik}c_{kj} \quad (16)$$

which equals:

$$-\sum_{k=1}^n b_{ik} + \sum_{k=1}^n b_{ik} \left(\sum_{j=1}^m c_{kj} \right) \quad (17)$$

which equals:

$$\sum_{k=1}^n [-b_{ik} + b_{ik} \left(\sum_{j=1}^m c_{kj} \right)] \quad (18)$$

or, equivalently:

$$\sum_{j=1}^m g_{ij} = \sum_{k=1}^n \left[\left(-1 + \sum_{j=1}^m c_{kj} \right) b_{ik} \right]. \quad (19)$$

As the authors have shown in [7], if $\det(Y_{LL}) \neq 0$, then every row of F_{LG} sums to one, i.e. $\sum_{j=1}^m c_{kj} = 1, \forall k = 1, 2, \dots, n$.

Thus, from (19), $\sum_{j=1}^m g_{ij} = 0$, and so every row of Y_{GGM} sums to zero. Similarly, if $\det(Y_{LL}) = 0$, then every row of F_{LG} sums approximately to one, i.e. $\sum_{j=1}^m c_{kj} \cong 1, \forall k = 1, 2, \dots, n$.

Thus, from (19), $\sum_{j=1}^m g_{ij} \cong 0$, and so every row of Y_{GGM} sums approximately to zero.

With equivalent steps if every row of Y_{bus} sums approximately to zero then every row of Y_{GGM} sums approximately to zero.

IV. MINIMIZING LOSSES

Taken together, (8) and the newly demonstrated zero row sum property of Y_{GGM} , and accordingly, its conjugate, imply that losses are minimized when generator voltages are homogeneous. As such, and disregarding machine limits, an optimal dispatch can be directly calculated by setting all V_G equal to the slack value, $1\angle 0$, and solving the resulting loadflow problem by the familiar iterative techniques. More insightfully, though, one can use the previously derived block matrix equations to see how this optimal dispatch relates to load current demands. Recall from (2):

$$I_G = K_{GL}I_L + Y_{GGM}V_G \quad (20)$$

Minimum loss conditions implies that V_G will be homogeneous, and so the second term of (20) reduces to zero. Therefore, we can infer the unconstrained generator dispatch that would realise this flat voltage profile:

$$I_G = K_{GL}I_L = S_G^{Opt} \quad (21)$$

The equivalence of I_G and S_G in these conditions is assured because generator voltages are all at the slack value. Assuming that load voltages are close to this value, we can further state that $S_L \approx I_L$. This interpretation of (21) in power terms shows how to use the K_{GL} and I_L to calculate an unconstrained generator dispatch that precludes circulating current losses. This explains the technique presented without proof in [1].

V. CONCLUDING EXAMPLE

Table I shows three possible dispatches for the 30 bus system. For context, the second row shows the dispatch achieved using the loss-minimizing optimization of [8]. The P_G^{Opt} dispatch achieves slightly lower losses than this. The largest row sum of this system's Y_{GGM} is $(-0.0028+0.0752j)$ i.e close to zero.

TABLE I
IEEE 30 BUS SYSTEM DISPATCHES (MW)

	G1	G2	G5	G8	G11	G13	Total
<i>Initial</i>	264.08	40.00	0.00	0.00	0.00	0.00	304.08
<i>ACOPF</i>	24.42	36.57	76.82	55.62	52.98	39.17	285.57
P_G^{Opt}	14.66	40.08	105.89	70.27	19.16	35.43	285.49

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